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NON-GAUSSIAN NOISE EFFECTS ON RELIABILITY OF MULTISTABLE SYSTEMS

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ABSTRACT

For certain types of compliant structures the designer must consider limit states associated with the onset of fluidelastic instability. These limit states may include bifurcations from motion in a safe region of phase space to chaotic motion with exits (jumps) out of the safe region. In practice such bifurcations occur in systems with noisy or stochastic excitations. For a wide class of dynamical systems, a fundamental connection between deterministic and stochastic chaos allows the application to stochastic systems of a necessary condition for the occurrence of chaos obtained by Melnikov for the deterministic case. We discuss the application of this condition to obtain probabilities that chaotic motions with jumps cannot occur in multistable systems excited by processes with tail-limited marginal distributions.

INTRODUCTION

The goal of structural reliability is to estimate probabilities of exceedance of limit states characterizing a structure's performance. For certain fluidelastic systems an important limit state is that associated with the onset of fluidelastic instability. This is defined as the occurrence of a bifurcation that causes an unfavorable change in the system's oscillatory form. An example of fluidelastic instability is the Hopf bifurcation occurring in structures such as the first Tacoma Narrows bridge. Below a critical wind velocity the dynamical system approximately representing such a bridge has a stable fixed point: a small disturbance of the system from its position of equilibrium will die out in time as the system is attracted by the stable fixed point. Beyond the bifurcation point (i.e., for velocities greater than the critical velocity) the stable fixed point becomes unstable, and the system experiences flutter motion, that is, it is attracted by a stable limit cycle surrounding the unstable fixed point.

Recent discoveries in the theory of deterministic dynamical systems include the existence of bifurcations from a periodic or quasiperiodic attractor to a chaotic attractor. For multistable systems such bifurcations can entail exits from a safe region (jumps) and can therefore be of concern from a structural reliability viewpoint. Similar jumps can occur in stochastic multistable systems. These phenomena are of potential interest in hydroelastic applications. A simple multistable hydroelastic system exhibiting chaotic behavior with jumps was studied experimentally and numerically by Simiu and Cook (1992). Other multistable hydroelastic systems with the potential for chaotic jumps are moored tankers and towed ships (Papoulias and Bernitsas, 1986; Bernitsas and Kekridis, 1988; Schellin, Jiang and Sharma, 1990; Jiang and Schellin, 1990). Chaotic motions were also found to occur in a numerical model of a compliant offshore tower studied by Thompson, Bokaian and Ghaffari (1984).

To illustrate the behavior of a multistable system we consider a simple bistable dynamical system consisting of a double potential well (Fig. 1a). In the absence of friction and forcing the system has two stable fixed points (centers), C and C', and one saddle point, O

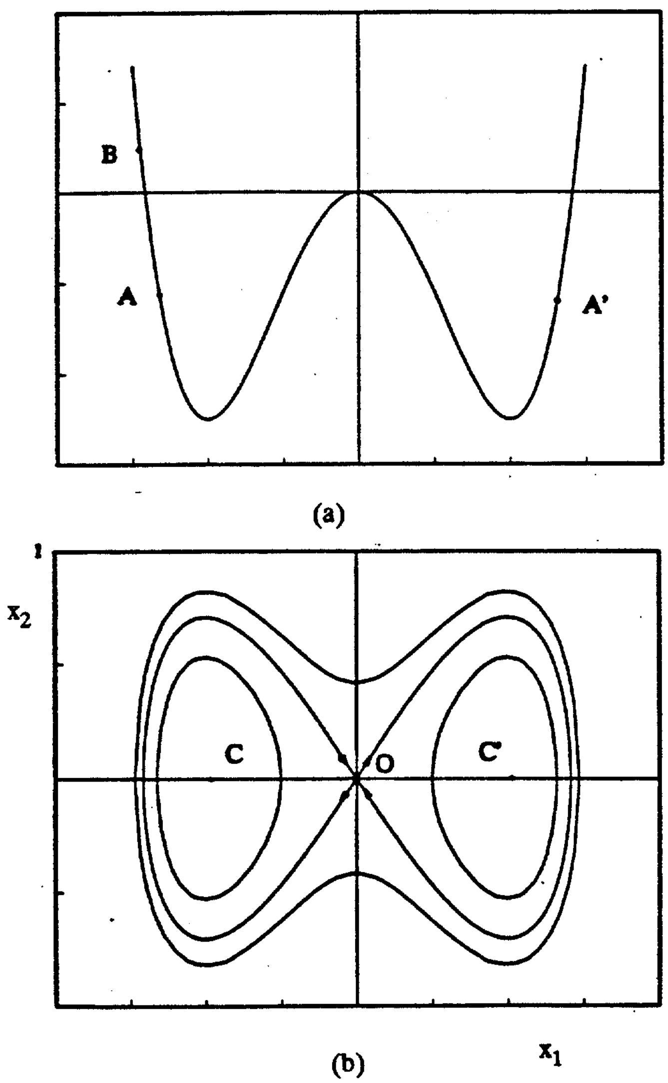

FIGURE 1. (a) POTENTIAL WELLS FOR BISTABLE SYSTEMS; (b) PHASE PLANE DIAGRAM

(Fig. 1b). A ball released from rest from point A would oscillate periodically on the closed curve centered around C in the phase plane diagram of Fig. 1b. (If the ball were released from point A' it would move periodically along a curve centered around C'.) If released from point B, the system would oscillate around O, as shown also in Fig. 1b. Motions that start inside (outside) the eight-shaped curve -- the separatrix -- shown in Fig. 1b stay inside (outside) it for all time. Motions starting on the separatrix will reach the saddle point O at time $t \to \infty$ and at time $t \to \infty$. The separatrix thus consists of a stable manifold that coincides with an unstable manifold, and is referred to as a homoclinic orbit.

If the system of Fig. 1 is perturbed (by the addition of sufficiently small friction and periodic forcing), it can be shown that, in a Poincare section, the saddle point will persist -- although slightly displaced. However, the homoclinic orbit will break, that is, the stable and unstable manifolds will no longer coincide. A distance can be defined between the perturbed system's stable and unstable manifolds. This distance is equal to the length of the segment defined by the intersections with the separated manifolds of a straight line L normal to the homoclinic orbit of the unperturbed system. To first order, that distance is proportional to the system's Melnikov function (see, e.g., Arrowsmith and Place, 1990, p. 172).

If the Melnikov function has simple zeros the stable and unstable manifolds of the perturbed system intersect transversely. Poincare sections through the stable and unstable manifolds then exhibit lobes, among which the so-called turnstile lobes have a privileged role (Wiggins, 1990). Under iteration by the Poincare map the turnstile lobes transport phase space from the interior to the exterior of the region bound by the pseudoseparatrix, and vice-versa. (The pseudoseparatrix is made up of segments defining lobes and is the counterpart in the perturbed system of the separatrix of Fig. 1b.) It can further be shown that this transport is chaotic, that is, owing to the folding of lobes under iteration the resulting map is topologically similar to the Smale horseshoe map. A necessary condition for the occurrence of deterministic chaos (and the attendant jumps) is that the Melnikov function have simple zeros. The Melnikov function -- and the corresponding necessary condition for the occurrence of chaos -- have been generalized to the case of quasiperiodic excitations by Beigie, Leonard and Wiggins (1992).

We now illustrate in Fig. 2 the possible effects of stochastic excitation (or noise) on the bistable system. Figure 2a shows a periodic motion induced in the harmonically forced, dissipative system. Figure 2b shows the same system in which a small amount of noise has been injected. The oscillation of Fig. 2b does not differ significantly from that of Fig. 2a, except for the presence of small "wiggles" induced by the noise. However, upon increasing the noise intensity, a bifurcation occurs. Past that bifurcation the motion, which was previously confined to a "safe" region, now

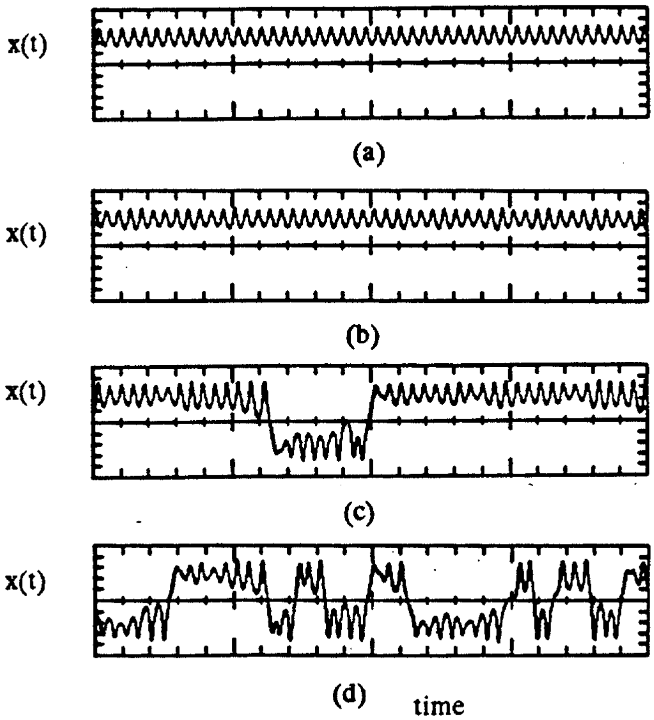

FIGURE 2. TIME HISTORIES: (a) WITHOUT NOISE; (b), (c), (d) WITH NOISE

crosses, chaotically, the potential barrier between the two regions associated with the potential wells. As the noise is increased further the chaotic motion exhibits more frequent jumps (smaller mean exit times) (Fig. 2d).

Chaotic time histories visually indistinguishable from Figs. 2c or 2d would be obtained if, instead of introducing noise in the system, we would keep the system deterministic but would increase the amplitude of the harmonic forcing by an appropriate amount. However, noise-induced (stochastic) chaos and deterministic chaos do not just look alike. Frey and Simiu (1993a) showed that the two types of chaos are in fact closely related mathematically. This fact allows the Melnikov criterion -- originally developed for deterministic chaos -- to be used in its generalized form to develop necessary conditions for the occurrence of stochastic chaos with jumps. In the case of Gaussian excitation the Melnikov criterion yields the trivial result that, with probability one, jumps will occur no matter how small the noise, although for very small noise the time between jumps may be expected to be very large (Frey and Simiu, 1993a).

The purpose of this paper is to estimate the reliability of a system with respect to exiting from a safe region in the technologically interesting case of random excitation with tail-limited marginal distributions. Until recently, for computational convenience and owing to insufficient knowledge, reliability computations were based in most cases on the assumption that probability distributions describing the forcing have infinite tails. In fact stochastic forces of interest in structural and offshore engineering have limited upper tails. This is due to limits on the magnitude of the extremes that are inherent in the relevant physical processes. Such limits were estimated recently for extreme wind speeds in certain climates by using statistical analyses based on novel "peaks over threshold" methods (Lechner, Leigh and Simiu, 1992).

Following a description of our approach, we illustrate its application by considering a simple dynamical system, the Duffing-Holmes oscillator. We then present our conclusions, including comments on the limitations of our approach and on needed future research.

RELIABILITY ESTIMATION

This section contains (1) a brief description of the generalized Melnikov function approach, (2) representations of stochastic processes with tail-limited marginal distribution that are consistent with the use of the Melnikov approach, (3) a derivation of the condition for non-occurrence of chaos in which one of these representations is used, and (4) an example of the use of this condition to estimate the reliability of the system with respect to the occurrence of chaotic motion with jumps.

Generalized Melnikov Function

We consider a second order system

$$\dot{x}_1 = x_2 \tag{1a}$$

$$\dot{x}_2 = f(x_1)+\epsilon[kx_2+\gamma\cos(\omega t+\phi)+\sigma X(t)] \tag{1b}$$

whose unperturbed flow has a homoclinic orbit $\{x_s(t), x_s(t)\}$.

The stochastic excitation $X(t)$ is approximated by the random process

$$X_m(t) = \sum_{i=1}^{m} \gamma_i \cos(\omega_i t+\phi_{io}) \tag{2}$$

where γ_i and ϕ_{io} (or ω_i and ϕ_{io}) are random variables. We assume X(t) is uniformly bounded and continuous. (This condition is satisfied if the marginal distribution of the process has limited tails.)

The expression for the Melnikov function is

$$M(\omega,\phi;\omega_1,\phi_{1o},...,\omega_m,\phi_{mo}) = -Ik + \gamma z(\omega,\phi) + \sigma\sum_{i=1}^{m}\gamma_i z(\omega_i,\phi_{io}) \quad (3)$$

where

$$I = \int_{-\infty}^{\infty} \dot{x}_s^2(t)dt \quad (4a)$$

$$z(\omega,\phi) = \int_{-\infty}^{\infty} \dot{x}_s(t)\cos(\omega t+\phi)dt \quad (4b)$$

$$z(\omega_i,\phi_{io}) = \int_{-\infty}^{\infty} \dot{x}_s(t)\cos(\omega_i t+\phi_{io})dt \quad (4c)$$

Chaotic motion cannot occur if

$$\gamma \max_{\phi}[z(\omega,\phi)] + \sigma \sum_{i=1}^{m} \max_{\phi_{oi}}[z(\omega_i,\phi_{io})] - Ik < 0 \quad (5)$$

Stochastic Process Representations

Let X(t) denote a non-Gaussian stochastic process with marginal distribution F. To apply the generalized Melnikov approach our first task is to seek approximations of X(t) in the form of Eq. 2.

Consider a stationary Gaussian process Y(t) with zero mean unit variance, covariance function $\rho_Y(t) = E[Y(t)Y(t+\tau)]$, and one-sided power spectral density function $g_Y(\Omega)$, $\Omega > 0$. Let Φ denote the distribution of the standard Gaussian variable. The process

$$X(t) = F^{-1}[\Phi(Y(t)] \quad (6)$$

has the marginal distribution F. By Eq. 6, to any specified spectral density $g_X(t)$ of X(t) there corresponds a covariance function $\rho_Y(t)$ and a spectral density $g_Y(\Omega)$. The latter must be used to simulate Y(t).

One possible approximation of X(t) in a form similar to Eq. 2 is obtained from its Fourier integral representation. First an approximation of the Gaussian process Y(t) is obtained by numerical simulation. Several simulation methods are available, including the Nyquist representation (Rice, 1954) and the Shinozuka (1971) representation. (as shown by Soong and Grigoriu (1992), the following representation is particularly efficient:

$$Y_N(t) = \sum_{k=1}^{N} R_k\sigma_k\cos(\Omega_k t+\theta_k) \quad (7)$$

where $\Omega_k = (k-1/2)\Delta\Omega$, $k = 1,2,..,N$ are equally spaced discrete frequencies in the frequency band $(0, N\Delta\Omega)$ in which most of the power of Y(t) is contained,

$$\sigma_k^2 = \int_{\Omega_k-\Delta\Omega/2}^{\Omega_k+\Delta\Omega/2} g(\Omega)d\Omega \quad (8)$$

denotes the variance associated with the harmonic k, $\{\theta_k\}$ are independent identically distributes (i.i.d.) variables uniformly distributed in $(0,2\pi)$,and $\{R_k\}$ are i.i.d. Rayleigh variables with the density $f(r) = r\exp(-r^2/2)$ for $r > 0$, and $f(r) = 0$ for $r < 0$. The approximation $Y_N(t)$ approaches Y(t) as $N \to \infty$.

Next, from $Y_N(t)$ we obtain corresponding values $X_N(t)$ using the transformation of Eq. 6. Finally, we represent $X_N(t)$ as an approximate Fourier integral

$$X_{m,N}(t) = (1/2\pi)\sum_{i=1}^{m} C(\Omega_i)\cos[\Omega_i t-\theta(\Omega_i)]\Delta\Omega_i \quad (9)$$

$$C(\Omega_i) = [A^2(\Omega_i) + B^2(\Omega_i)]^{1/2} \quad (10)$$

$$\theta(\Omega_i) = \tan^{-1}[B(\Omega_i)/A(\Omega_i)] \quad (11)$$

$$A(\Omega_i) = \sum_{k=1}^{M} X_N(t_k)\cos\Omega_i t_k\Delta t_k \quad (12)$$

$$B(\Omega_i) = \sum_{k=1}^{M} X_N(t_k)\sin\Omega_i t_k\Delta t_k \quad (13)$$

Since Eq. 9 has the same form as Eq. 2 it can be used to obtain the sum of the left hand side of Eq. 5 for a sufficiently large number of realizations of $Y_N(t)$. Once this is done the probability that Eq. 5 holds can be estimated for any given intensity σ. This brute force procedure is seen to require a fairly large, though not prohibitive, amount of computation.

We now examine the following alternative where the nonlinear transformation in Eq. 6 is approximated by a polynomial of degree n. This can be done to any accuracy in a bounded interval. Let

$$X_{n,N}(t) = p_n(Y_N(t)) \tag{14}$$

be a polynomial approximation of X(t) in Eq. 6 based on the representation $Y_N(t)$ of Y(t), in which

$$p_n(y) = \sum_{j=0}^{n} a_j y^j \tag{15}$$

The non-Gaussian noise $X_{n,N}(t)$ follows the marginal distribution F approximately and can be expressed as a linear combination of harmonics with random amplitudes. This statement is based on Eqs. 7, 14, 15 and the equality

$$\prod_{i=1}^{n} \cos\beta_i = (1/2^{n-1}) \sum_{p_2,\ldots,p_n=0,1} \cos[\beta_1 + \sum_{j=2}^{n}(-1)^{pj}\beta_j] \tag{16}$$

which shows that products of cosine functions can be expressed as sums of cosine functions. We use Eq. 14 as our approximate representation of the tail-limited stochastic excitation.

Equations 7 and 14 yield

$$X_{n,N}(t) = a_o + a_1 \sum_{k=1}^{N} \sigma_k R_k \cos(\Omega_k t + \theta_k)$$

$$+ a_2 \sum_{j,k=1}^{N} \sigma_j \sigma_k R_j R_k \cos(\Omega_j t + \theta_j)\cos(\Omega_k t + \theta_k) + ..$$

$$+ a_n \sum_{k_1,\ldots,k_n=1}^{N} \prod_{i=1}^{n} [\sigma_{ki} R_{ki} \cos(\Omega_{ki} t + \theta_{ki})] \tag{17}$$

From Eqs. 16 and 17

$$X_{n,N}(t) = a_o + a_1 \sum_{k=1}^{N} \sigma_k R_k \cos(\Omega_k t + \theta_k)$$

$$+ \frac{1}{2} a_2 \sum_{j,k=1}^{N} \sigma_j \sigma_k R_j R_k \{\cos[(\Omega_j + \Omega_k)t + (\theta_j + \theta_k)] +$$

$$\cos[(\Omega_j - \Omega_k)t + (\theta_j - \theta_k)]\} + ..$$

$$+ \frac{1}{2^{n-1}} a_n \sum_{k_1,\ldots,k_n}^{N} \prod_{i=1}^{n} (\sigma_{ki} R_{ki})$$

$$\sum_{p_2,\ldots,p_n=0,1} \cos[(\Omega_{k1} + \Sigma(-1)^{pj}\Omega_{kj})t + (\theta_{k1} + \Sigma(-1)^{pj}\theta_{kj})] \tag{18}$$

The approximation of the stochastic process has now the form of Eq. 2.

Approximate Condition for Non-Occurrence of Chaos

We now show, for the case of the polynomial approximation, how Eqs. 2 (for which we substitute Eq. 18), and Eqs. 3, 4 and 5 are used to obtain the approximation of the condition guaranteeing that chaos does not occur. We have

$$M^* = \alpha + \sigma[a_o\alpha_o + a_1 \sum_{k=1}^{N} \alpha_k R_k + a_2 \sum_{k=1}^{N} \alpha_{j,k} R_j R_k + ..$$

$$+ a_n \sum_{k_1,\ldots,k_n=1}^{N} \alpha_{k1,\ldots,kn} \prod_{i=1}^{n} R_{ki}] < 0 \tag{19}$$

where

$$\alpha = - Ik + \gamma \max[z(\omega,\phi)] \tag{20}$$

$$\alpha_o = z(0,0) \tag{21}$$

$$\alpha_k = \max[z(\Omega_k, \theta_k)] \tag{22}$$

$$\alpha_{j,k} = (1/2)\, \sigma_j \sigma_k \{\max[z(\Omega_j + \Omega_k, \theta_j + \theta_k)] + \max[z(\Omega_j - \Omega_k, \theta_j - \theta_k)]\} \tag{23}$$

$$\alpha_{k1,..kn} = (1/2^{n-1}) \prod_{i=1}^{n} \sigma_{ki} \sum_{p_2,\ldots,p_n=0,1} \max\{z[\Omega_{k1} + \sum_{j=2}^{n}(-1)^{pj}\Omega_{kj}, \theta_{k1} + \sum_{j=2}^{n}(-1)^{pj}\theta_{kj}]\} \tag{24}$$

where $\max[z(\Omega,\theta)]$ denotes the maximum of the function z over all possible values of θ.

Example. We illustrate the procedure based on polynomial approximations for the Duffing-Holmes oscillator, for which $f(x_1) = x_1 - x_1^3$. The coordinates of the homoclinic orbit for the unperturbed system are

$$x_s(t) = (2)^{1/2}\text{secht}$$

$$\dot{x}_s(t) = (2)^{1/2}\text{secht tanht}$$

We use the righthand (+) orbit in our calculations. The same results are obtained for the lefthand orbit. In this case $I = -4/3$, $z(0,0)$ vanishes and the function $z(\Omega,\theta) = (2)^{1/2}\pi\Omega\text{sech}(\pi\Omega/2)\cos\theta$ (Arrowsmith and Place, 1990), so that $\max[z(\Omega,\theta)] = (2)^{1/2}\pi\Omega\text{sech}(\pi\Omega/2)$. We assume in our example $\epsilon k = 0.15$, $\epsilon\gamma = 0.15$, and the stochastic process

$$X(t) = a + (b\text{-}a)\ \Phi[Y(t)]$$

where $a = -1$, $b = 1$ (i.e., the distribution of X is $F(X) = (X - a)/(b - a)$, which corresponds for our choice of parameters a and b to $F(X) = 0$ for $X < -1$, $F(X) = 0$ for $X > 1$, and a linear variation of F(X) between these limits). The process Y(t) is assumed to be Gaussian with zero mean, unit variance, and spectral density $g(\Omega) = 0.2$ $(0 < \Omega < 5)$, $g(\Omega) = 0$ $(\Omega < 0$ and $\Omega > 5)$. We illustrate the approach based on polynomial approximation.

$\Phi(y)$ may be approximated by a polynomial in odd powers of y. If the approximating polynomial is of degree three, that is, if

$$\Phi(y) \approx a_o + a_1 y + a_3 y^3$$

we must have $a_o = 0.5$ and least squares fitting yields $a_1 = 0.329652222$, $a_3 = -0.018084395$. It follows that

$$X(t) \approx -0.036168790\ Y(t)^3 + 0.659304444\ Y(t)$$

The results obtained by using Eq. 7 with $N = 50$ and 1000 samples are listed in Table 1.

To a first approximation, it can be stated that, with probability one, the system will not experience chaos provided that $\sigma < 0.25$. Our experience with similar polynomial approximations suggests that using a higher degree polynomial does not affect the estimated probabilities significantly, especially for large probabilities, which are of primary concern in practice.

TABLE 1. ESTIMATED PROBABILITY THAT APPROXIMATE MELNIKOV FUNCTION M* HAS NO SIMPLE ZEROS

σ	<0.25	0.30	0.35	0.40	0.45	0.50
P(M*<0)	1.000	0.982	0.685	0.193	0.300	0.006

DISCUSSION AND CONCLUSIONS

In this paper we used the generalized Melnikov approach to estimate the reliability with respect to the occurrence of jumps of a wide class of multistable systems subjected to additive noise. We dealt with systems whose stochastic excitations have tail-limited probability distributions. This case is of interest because physical constraints limit the magnitude of random variates of interest in offshore engineering (e.g., waves). We considered two possible representations of the random excitation: (1) a Fourier integral representation of the non-Gaussian process, and (2) a polynomial approximation.

The systems considered in this exploratory work have one degree of freedom and unforced, frictionless counterparts with homoclinic or heteroclinic orbits. For some multistable systems the identification of these orbits is straightforward. However, for some systems encountered in offshore engineering such orbits may be difficult if not impossible to identify. The extent to which this would limit the applicability of our approach remains to be ascertained. Also, extending our approach to multi-degree-of-freedom systems is not a trivial task, even though applications of the Melnikov approach to such systems have been reported (see, e.g., Allen, Samelson and Newberger, 1991).

The noise was assumed to be additive. However, the extension to multiplicative noise is straightforward (Frey and Simiu, 1993b). Also, even though Melnikov theory is based on the assumption that the perturbations are small, it is in practice applicable to systems with fairly large perturbations (see, e.g., Moon, 1987). Our approach is therefore not restricted to small forcing and friction.

To our knowledge, the Melnikov-based approach presented here provides the only available means for developing a computable criterion that guarantees the non-occurrence of jumps in the types of systems we investigated. Work on its extension to more complex systems and on reducing its current limitations is currently in progress.

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